Definition 20. The aggregate operation $Min_{A,R}$ by the attribute A of the finite table on scheme R, $A \in R$, is a unary parametric operation of the form $Min_{A,R}: \Psi(R) \to \Psi(\{A\})$,
$Min_{A,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle A, Min(\alpha_A)\rangle\}^1\right\}, \{A\} \right\rangle$, where $\langle \psi, R \rangle \in \Psi(R)$. The $Min(\alpha_A)$ function is applied to a column with attribute A in the table $\langle \psi, R \rangle$, the result obtained is the minimum value among values of α_A. In addition, $NULL$ values don't undertake in attention.

Thus, $Min: 2_m^{\boldsymbol{D}} \to \boldsymbol{D}$,

$$Min(\alpha_A) = \begin{cases} NULL \text{ if } \Theta(\alpha_A) = \varnothing; \\ NULL \text{ if } \Theta(\alpha_A) = \{NULL\}; \\ \min\{d \mid d \in \Theta(\alpha_A) \setminus \{NULL\}\} \text{ if } \Theta(\alpha_A) \setminus \{NULL\} \neq \varnothing. \end{cases}$$

We have $Min(\varnothing_m) = NULL$, $Min(\{NULL^n\}) = NULL$, $Min(\{d_1^{n_1}, \ldots, d_k^{n_k}\}) = \min\{d_1, \ldots, d_k\}$ if all elements d_i, $i = \overline{1,k}$, differ from $NULL$.

In the case of the empty table $\langle \psi_\varnothing, R \rangle$ we have $Min_{A,R}(\langle \psi_\varnothing, R \rangle) = \left\langle \left\{\{\langle A, NULL\rangle\}^1\right\}, \{A\} \right\rangle$, here $\psi_\varnothing = \varnothing_m$.

Example 8. Let $\langle \psi, R \rangle$ be the Table 6. Then
$Min_{A,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle A,2\rangle\}^1\right\}, \{A\} \right\rangle$,
$Min_{B,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle B,0\rangle\}^1\right\}, \{B\} \right\rangle$,
$Min_{C,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle C,1\rangle\}^1\right\}, \{C\} \right\rangle$.

Definition 21. The aggregate operation $Max_{A,R}$ by the attribute A of the finite table on scheme R, $A \in R$, is a unary parametric operation of the form $Max_{A,R}: \Psi(R) \to \Psi(\{A\})$,
$Max_{A,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle A, Max(\alpha_A)\rangle\}^1\right\}, \{A\} \right\rangle$, where $\langle \psi, R \rangle \in \Psi(R)$. The $Max(\alpha_A)$ function is applied to a column with attribute A in the table $\langle \psi, R \rangle$, the result obtained is the maximum value among values of α_A. In addition, $NULL$ values don't undertake in attention.

Thus, $Max: 2_m^{\boldsymbol{D}} \to \boldsymbol{D}$,

$$Max(\alpha_A) = \begin{cases} NULL \text{ if } \Theta(\alpha_A) = \varnothing; \\ NULL \text{ if } \Theta(\alpha_A) = \{NULL\}; \\ \max\{d \mid d \in \Theta(\alpha_A) \setminus \{NULL\}\} \text{ if } \Theta(\alpha_A) \setminus \{NULL\} \neq \varnothing. \end{cases}$$

We have $Max(\varnothing_m) = NULL$, $Max(\{NULL^n\}) = NULL$, $Max(\{d_1^{n_1}, \ldots, d_k^{n_k}\}) = \max\{d_1, \ldots, d_k\}$ if all elements d_i, $i = \overline{1,k}$, differ from $NULL$.

In the case of the empty table $\langle \psi_\varnothing, R \rangle$ we have $Max_{A,R}(\langle \psi_\varnothing, R \rangle) = \left\langle \left\{\{\langle A, NULL\rangle\}^1\right\}, \{A\} \right\rangle$, here $\psi_\varnothing = \varnothing_m$.

Example 9. Let $\langle \psi, R \rangle$ be the Table 6. Then
$Max_{A,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle A,2\rangle^1\}\right\}, \{A\} \right\rangle$,
$Max_{B,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle B,3\rangle\}^1\right\}, \{B\} \right\rangle$,
$Max_{C,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle C,3\rangle\}^1\right\}, \{C\} \right\rangle$.

Definition 22. The aggregate operation $Count_{A,R}$ by the attribute A of the finite table on scheme R, $A \in R$, is a unary parametric operation of the form $Count_{A,R}: \Psi(R) \to \Psi(\{A\})$,
$Count_{A,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle A, Count(\alpha_A)\rangle\}^1\right\}, \{A\} \right\rangle$, where $\langle \psi, R \rangle \in \Psi(R)$. The $Count(\alpha_A)$ function is applied to a column with attribute A in the table $\langle \psi, R \rangle$, the result obtained is the count of all values of α_A which differ from $NULL$.

Thus, $Count: 2_m^{\boldsymbol{D}} \to \{0,1,2,\ldots\}$, $Count(\alpha_A) = \sum_{d \in \Theta(\alpha_A) \setminus \{NULL\}} \alpha_A(d).$ Put by definition that the sum of an empty set of elements is equal to zero.

So, we have $Count(\varnothing_m) = 0$, $Count(\{NULL^n\}) = 0$, $Count(\{d_1^{n_1}, \ldots, d_k^{n_k}\}) = n_1 + \ldots + n_k$ if all elements d_i, $i = \overline{1,k}$, differ from $NULL$.

In the case of the empty table $\langle \psi_\varnothing, R \rangle$ we have $Count_{A,R}(\langle \psi_\varnothing, R \rangle) = \left\langle \left\{\{\langle A,0\rangle\}^1\right\}, \{A\} \right\rangle$, here $\psi_\varnothing = \varnothing_m$.

Example 10. Let $\langle \psi, R \rangle$ be the Table 6. Then
$Count_{A,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle A,4\rangle\}^1\right\}, \{A\} \right\rangle$,
$Count_{B,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle B,5\rangle\}^1\right\}, \{B\} \right\rangle$,
$Count_{C,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle C,4\rangle\}^1\right\}, \{C\} \right\rangle$.

We assume that a numerical subset Num of the universal domain $\boldsymbol{D}$ is closed under the (partial operation) division operation $/ : Num \times Num \xrightarrow{\sim} Num$. We will determine the division operation so that when the first argument is equal to $NULL$ the function accepts value $NULL$.

Definition 23. The aggregate operation $Avg_{A,R}$ by the attribute A of the finite table on scheme R, $A \in R$, is a unary parametric operation of the form $Avg_{A,R}: \Psi(R) \to \Psi(\{A\})$,
$Avg_{A,R}(\langle \psi, R \rangle) = \left\langle \left\{\{\langle A, Avg(\alpha_A)\rangle\}^1\right\}, \{A\} \right\rangle$, where

$\langle\psi, R\rangle \in \Psi(R)$. The $Avg(\alpha_A)$ function is applied to a column with attribute A in the table $\langle\psi, R\rangle$, the result obtained is the arithmetic mean of values in α_A which differ from $NULL$. Thus, $Avg: 2_m^{Num} \to Num$ and $Avg(\alpha_A) = \frac{Sum(\alpha_A)}{Count(\alpha_A)}$. We have $Avg(\varnothing_m) = \frac{Sum(\varnothing_m)}{Count(\varnothing_m)} = \frac{NULL}{0} = NULL$, $Avg(\{NULL^n\}) = \frac{Sum(\{NULL^n\})}{Count(\{NULL^n\})} = NULL$, $Avg(\{d_1^{n_1}, \ldots, d_k^{n_k}\}) = \frac{Sum(\{d_1^{n_1}, \ldots, d_k^{n_k}\})}{Count(\{d_1^{n_1}, \ldots, d_k^{n_k}\})} =$ $= \frac{\sum_{i=1}^{k} d_i n_i}{(n_1 + \ldots + n_k)}$ if all elements d_i, $i = \overline{1,k}$, differ from $NULL$.

In the case of the empty table $\langle\psi_\varnothing, R\rangle$ we have $Avg_{A,R}(\langle\psi_\varnothing, R\rangle) = \langle\{\{\langle A, NULL\rangle\}\}, \{A\}\rangle$, here $\psi_\varnothing = \varnothing_m$.

Example 11. Let $\langle\psi, R\rangle$ be the Table 6. Then

$$Avg_{A,R}(\langle\psi, R\rangle) = \left\langle\left\{\{\langle A, 2\rangle\}^1\right\}, \{A\}\right\rangle,$$

$$Avg_{B,R}(\langle\psi, R\rangle) = \left\langle\left\{\left\{\left\langle B, \frac{6}{5}\right\rangle\right\}^1\right\}, \{B\}\right\rangle,$$

$$Avg_{C,R}(\langle\psi, R\rangle) = \left\langle\left\{\left\{\left\langle C, \frac{6}{4}\right\rangle\right\}^1\right\}, \{C\}\right\rangle.$$

Definition 24. The aggregate operation $Count_{A,R}(*)$ by the attribute A of the finite table on scheme R, $A \in R$, is a unary parametric operation of the form $Count_{A,R}(*): \Psi(R) \to \Psi(\{A\})$, $Count_{A,R}(*)(\langle\psi, R\rangle) = \left\langle\left\{\left\{\langle A, \|\psi\|\rangle\right\}^1\right\}, \{A\}\right\rangle$, where $\langle\psi, R\rangle \in \Psi(R)$, and $\|\psi\|$ is the rank of the multiset ψ.

The operation $Count_{A,R}(*)$ finds the number of tuples in the table $\langle\psi, R\rangle$.

In the case of an empty table $\langle\psi_\varnothing, R\rangle$ we have $Count_{A,R}(*)(\langle\psi_\varnothing, R\rangle) = \left\langle\left\{\{\langle A, \|\varnothing_m\|\rangle\}^1\right\}, \{A\}\right\rangle =$ $= \left\langle\left\{\{\langle A, 0\rangle\}^1\right\}, \{A\}\right\rangle$, here $\psi_\varnothing = \varnothing_m$.

Example 12. Let $\langle\psi, R\rangle$ be the Table 6. Then

$$Count_{A,R}(*)(\langle\psi, R\rangle) = \left\langle\left\{\{\langle A, 5\rangle\}^1\right\}, \{A\}\right\rangle,$$

$$Count_{B,R}(*)(\langle\psi, R\rangle) = \left\langle\left\{\{\langle B, 5\rangle\}^1\right\}, \{B\}\right\rangle,$$

$$Count_{C,R}(*)(\langle\psi, R\rangle) = \left\langle\left\{\{\langle C, 5\rangle\}^1\right\}, \{C\}\right\rangle.$$

5. Conclusions

In this paper the multiset table algebra is considered. The concept of the table is specified, using concept of a multiset. The signature of the multiset table algebra is filled up with new operations such as inner and outer join, semijoin and aggregate operations. For each operations are defined a basis of the resulting table and number of duplicates of every tuple. The special element NULL is inserted in the universal domain for a define of outer operations.

It should also be noted that a parameter of aggregate operations is not necessarily only a single attribute; it also can be some function of the tuples.

6. References

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