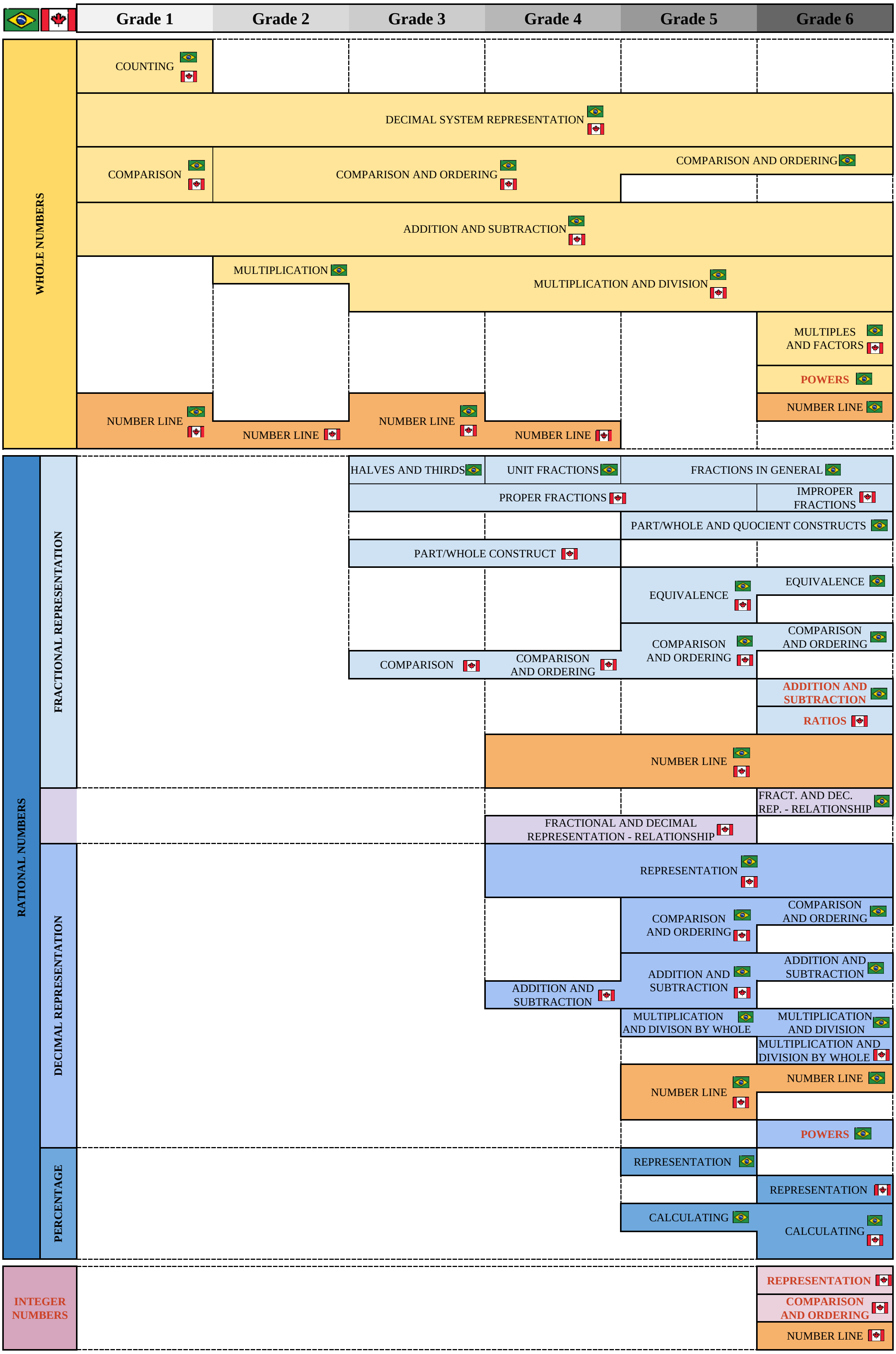

Figure 2. Curricular trajectory (numbers, grades 1 to 6)

5. The curricular trajectory and teacher's practice: the teaching of fractions

One of the goals of analyzing school mathematics curricula based on curricular documents from different countries is to promote reflection on teaching processes and contribute to teachers' practice. Ultimately, the aim is to contribute to students' mathematics learning. With this goal in mind, this study provides mathematics teachers with a visual tool, the *curricular trajectory*, which draws attention to content- and teaching-related aspects of mathematics education. As a result, teachers can question, analyze, and evaluate different teaching proposals related to the same mathematical content. We understand that this is an exercise that contributes to the development of teachers' own knowledge, the knowledge of mathematics for teaching ([3], see Figure 3). The knowledge about the curriculum is an integral part of the knowledge of mathematics for teaching ([3] [22]).

> By "mathematical knowledge for teaching," we mean the mathematical knowledge needed to carry out the work of teaching mathematics. Important to note here is that our definition begins with teaching, not teachers. It is concerned with the tasks involved in teaching and the mathematical demands of these tasks. Because teaching involves showing students how to solve problems, answering students' questions, and checking students' work, it demands an understanding of the content of the school curriculum. [3] (p.21)

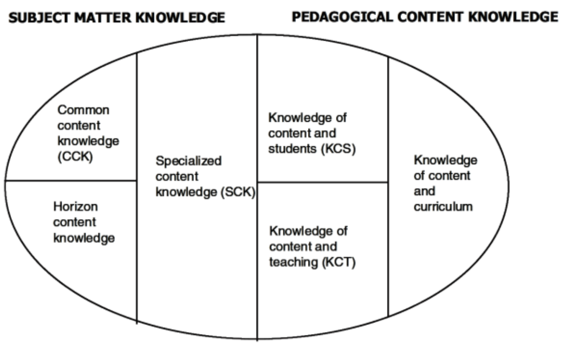

Figure 3. Domains of mathematical knowledge for teaching [3] (p. 403)

To exemplify a few possibilities for reflection, provided by the *curricular trajectory*, we can look at rational numbers. The study of rational numbers can be investigated starting from the introduction to fractions until the study of percentages. The *curricular trajectory* suggests that the initial approach to fractions in both countries is quite different. We will focus our analysis on this fraction initial approach – whose *structuring elements* are portrayed in Table 3. The *curricular trajectory* indicates *parities* and *contrasts* between the Brazilian and the Canadian approaches that raise questions. Some of these questions are discussed below.

Fractions arise from the need to register quantities smaller than the unit. However, they also register quantities greater than 1. The Brazilian dimension of the *curricular trajectory* shows the introduction of fractions, in grade 3, starting from halves and thirds, followed by unit fractions in grade 4, and fractions in general in grades 5 and 6 (see Table 3). In other words, the Brazilian document recommends the approach based on unitary fractions, without suggesting the differentiation between proper and improper fractions. A fraction could be expressed by the "sum" of unit fractions, as proposed in [23], for example. The Canadian dimension, on the other hand, recommends starting the teaching of fractions by proper fractions, in grades 3, 4 and 5. Improper fractions are addressed in grade 6 (see Table 3). The nominal difference for fractions smaller and larger than the unit is not even highlighted in the Brazilian document. Hence, at least one contrast is in evidence and raises the question: *"Why or why not differentiate between proper and improper fractions in the teaching of fractions?"*

The part/whole construct is strongly associated with the teaching of fractions [24]. The analysis of the *curricular trajectory* points to a *contrast* that leads to a reflection on when such a concept should be discussed. The BNCC explicitly proposes this discussion, in grade 5, two years after the beginning of the teaching of fractions (see Table 3). The CCF, on the other hand, brings the part/whole construct in the first year of teaching about fractions, in grade 3 (see Table 3). This observation is associated with another question that is intrinsic to the knowledge of mathematics for teaching: *"What is the relevance of and the relationship between the part-whole construct and the other constructs associated with fractions?"* For Behr and his collaborators, as the scheme shown in Figure 4 illustrates, the concept of fractions involves five distinct but interrelated constructs: Part/Whole, Ratio, Operator, Quotient and Measure. From this perspective, the part/whole construct has a primary value, being fundamental for the composition of the other meanings attributed to fractions. Regardless of its relevance, it is up to the teacher to ensure that it does not overlap with others. Learning fractions should address all five constructs. It should be noted that the Brazilian curriculum document only proposes two of these constructs (part/whole and quotient) in grades 5 and 6 (see Table 3). The Canadian document highlights only the part/whole construct in grades 3 and 4 (see Table 3), although it deals with ratios in grade 6 and alludes to fractions in the achievement indicators; "Express a given ratio in multiple forms, such as 3:5, $\frac{3}{5}$, or 3 to 5" [6] (p.111).

Equivalence is certainly another elementary theme in the construction of the concept of fractions. However, it is not an easy concept, it requires abstraction. It requires understanding that one same quantity, related to a given unit, can have different representations.

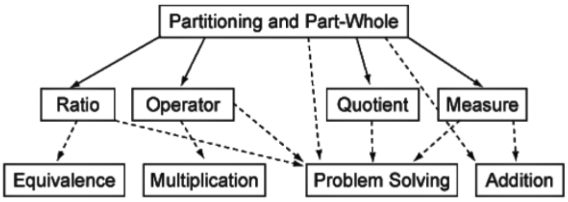

Figure 4. A conceptual scheme for instruction on rational numbers [24] (p. 100)

We could ask, *"At what schooling stage should this notion of equivalence be introduced?"*. The *curricular trajectory* indicates that both the curricular guidelines from Brazil and Canada propose this introduction at grade 5, two years after the introduction of the concept of fractions (see Table 3). However, if the *structuring elements* of the curricular documents are examined in greater depth, we can see that this *parity* has the potential to promote reflections inherent to teachers' knowledge. The BNCC presents equivalence as a grade 5 skill and associates it to the comparison and ordering of fractions (see Table 3). A CCF specific outcome also deals with this association; however, it highlights the need for students to understand that equivalent fractions belong to a set of fractions that represent the same amount (see Table 3). Certainly, this is a learning objective.

Likewise the equivalent fractions approach, the teaching of comparison and ordering of fractions can be questioned. When looking at the *curricular trajectory*, we notice that the CCF suggests the comparison of fractions in grade 3, together with the initial approach to fractions (see Table 3). This discussion is expanded in grade 4 and includes ordering as well. Both contents are discussed until grade 5 in the Canadian guidelines. On the other hand, the BNCC suggests the introduction of comparison and ordering of fractions in grades 5 and 6, together with the study of equivalent fractions (see Table 3). It is evident that there is a mismatch between the two curricular guidelines when it comes to the starting point of the teaching of comparison and ordering of fractions: grade 3 in the CCF and grade 5 in the BNCC. In addition to this chronological mismatch, we can also ask: *"What is the potential contribution or limitation of the teaching of comparison and ordering of fractions linked to the equivalence of fractions?"* It is possible, and it should be expected, that a student decides, for example, that $\frac{1}{6}$ is greater than $\frac{1}{8}$ just by observing the equipartition of the unit. This result can support the recognition that $\frac{5}{6}$ is, therefore, greater than $\frac{5}{8}$. It can also be observed (with or without the support of visual representation) that $\frac{5}{6}$ is smaller than $\frac{7}{8}$ since the former lacks $\frac{1}{6}$ to complete the unit and the latter $\frac{1}{8}$ (which is smaller than $\frac{1}{6}$). Without disregarding the relevance of equivalence to the development of the processes of comparison and ordering of fractions, procedures such as those described above certainly contribute to the development of less mechanized reasoning.

Another aspect that can be analyzed is the introduction of basic operations (addition, subtraction, multiplication, and division) involving fractions. The BNCC determines that the teaching of addition and subtraction of fractions should begin at grade 6 (see Table 3). The CCF does not mention this content on or before grade 6 (see Table 3). It is also observed that none of the documents proposes the teaching of multiplication and division of fractions in the period analyzed. The reflection is immediate: "*At what schooling stage should the teaching of basic operations involving fractions begin?*". It is worth questioning what cognitive demands each of these operations imposes [25]. For example, Son [25], based on the observation of problems proposed in textbooks and curriculum guidelines from Korea and the USA, examines the opportunities for learning the addition and subtraction of fractions at the grade school level grounded on students' cognition. It is a fact that the addition and subtraction of fractions can be supported by the conceptual understanding of these operations in the context of whole numbers: joining, separating, and comparing [17] [26]. However, this conceptual extension does not match the calculation procedures. It is necessary to understand how to add (or subtract) fractions with the same denominators and with different denominators. Inquiries like these can lead to reflections about student's cognitive and abstract development when it comes to fractions, and to reflections about the teaching of fractions and its curricular distribution.

By observing the *curricular trajectory*, other questions may emerge when seeking to extrapolate the teaching of fractions and relating it to other content specific to the teaching of rational numbers. For example, we observe that the Brazilian document proposes the relationship between fractions and decimal representation only in grade 6 (see Figure 2). The Canadian document, in contrast, proposes this relationship in grade 4 (see Figure 2). Still on rational numbers, the *curricular trajectory* reveals that the teaching of percentage starts in Brazil in grade 5, while in Canada only in grade 6 (see Figure 2); however, the *curricular trajectory* does not identify which representations (percentual, fractional, decimal) are explored and how such representations should be related. The analysis of the *curricular trajectory* also enables questions beyond the teaching of rational numbers. We observe, for example, that the number line is explored across all number sets addressed from grade 1 to grade 6. The number line supports the teaching of whole numbers, rational numbers, and integers (in the case of the Canadian curriculum guidelines). Particularly, it is worth exploring the role of the number line in the teaching and learning of fractions. Investigating and understanding the underpinnings behind the different approaches discussed above can potentially promote new reflections, studies, questionings, and developments on the teaching of numbers, especially in what concerns teachers' practice.

Table 3. Structuring elements – fractions

	BNCC [5]	**CCF [6]**
GRADE 3	Skills (EF03MA09) Associate the quotient of a division with zero remainder of a natural number by 2, 3, 4, 5 and 10 to the ideas of halves, thirds, fourths, fifths and tenths. (p. 287)	Specific Outcome 13. Demonstrate an understanding of fractions by: ✓ explaining that a fraction represents a part of a whole ✓ describing situations in which fractions are used ✓ comparing fractions of the same whole with like denominators. (p. 78)
GRADE 4	Skills (EF04MA09) Recognize the most common unit fractions (1/2, 1/3, 1/4, 1/5, 1/10 and 1/100) as measurement units smaller than one unit, using the number line as a resource. (p. 291)	Specific Outcome 8. Demonstrate an understanding of fractions less than or equal to one by using concrete and pictorial representations to: ✓ name and record fractions for the parts of a whole or a set ✓ compare and order fractions ✓ model and explain that for different wholes, two identical fractions may not represent the same quantity ✓ provide examples of where fractions are used. (p. 89)
GRADE 5	Skills (EF05MA03) Identify and represent fractions (smaller and larger than unity), associating them to the result of a division or to the idea of part of a whole, using the number line as a resource. (p. 295) (EF05MA04) Identify equivalent fractions. (p. 295) (EF05MA05) Compare and order positive rational numbers (fractional and decimal representations), relating them to points on the number line. (p. 295)	Specific Outcome 7. Demonstrate an understanding of fractions by using concrete and pictorial representations to: ✓ create sets of equivalent fractions ✓ compare fractions with like and unlike denominators. (p. 99)
GRADE 6	Skills (EF06MA07) Understand, compare and order fractions associated with the ideas of parts of integers and the result of a division, identifying equivalent fractions. (p. 301) (EF06MA08) Recognize that positive rational numbers can be expressed in fractional and decimal forms, establish relationships between these representations, moving from one representation to another, and relate them to points on the number line. (p. 301) (EF06MA09) Solve and elaborate problems involving the calculation of the fraction of a quantity and whose result is a whole number, with and without the use of a calculator. (p. 301) (EF06MA10) Solve and elaborate problems involving addition or subtraction with positive rational numbers in the fractional representation. (p. 301)	Specific Outcome 4. Relate improper fractions to mixed numbers. (p. 110)

6. Conclusion

"Research that looks across countries can provide a sharper picture of what matters in instruction aimed at developing [mathematical] proficiency" [17] (p. 358). We argue that relating broad-reaching curricular references, such as the BNCC [5] and the CCF [6], can contribute (i) to the evaluation and development of such curricular references, (ii) to the development of materials and textbooks, and (iii) to guide teachers' training and practice. Ultimately, focused on improving students' mathematical learning. In this sense, this study, which focuses on the teaching of numbers, proposes a reflective discussion led by questions about the teaching of fractions. The underlined issues are based on the observation of the *curricular trajectory* – a multidimensional graphic scheme, product of this study, that organizes curricular emerging *parities* and *contrasts*, grounded on the combined reading of Brazilian and Canadian curricular documents.

The *curricular trajectory* surpasses its conclusive nature and offers reflections that have the potential to unfold new research questions and developments of this study. We understand that the *curricular trajectory* can be appreciated as a vertex of a graph. One can look backward, seeking to deepen several issues that, even though were observed in this study, are not evident in the direct reading of the infographic. Or one can look forward, raising new questions, pointing to or inspiring future studies, or even serving as a reference to support various investigations. The *curricular trajectory* can also extrapolate its original focus on the intended curriculum and contribute to research investigations focused on the implemented curriculum. In other words, the *curricular trajectory* can foster not only curriculum-related analysis, but also didactic-pedagogical discussions, and the investigation of textbooks or other teaching materials. Such a change in focus enables a shift from the intended to the implemented curriculum, further contributing to curricular discussions in the mathematics education community.

7. References

[1] Li, Y. and Lappan, G. (2014) Mathematics Curriculum in School Education, Springer, Netherlands.

[2] Shimizu, Y. and Vithal, R. (2018) School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, University of Tsukuba, Tsukuba.

[3] Ball, D., Thames, M. H. and Phelps, G. (2008) "Content knowledge for teaching: What makes it special?", Journal of Teacher Education, 59(5), pp. 389-407.

[4] UNESCO (2016) Glossary of Curriculum Terminology, UNESCO International Bureau of Education, Paris.

[5] Brasil (2017) Base Nacional Comum Curricular, Ministério da Educação, Brazil.

[6] Alberta Education (2006) The Common Curricular Framework for K-9 Mathematics, Alberta Education, Alberta.

[7] Garnica, A. V. M. (2014) "Brief Considerations on Educational Directives and Public Policies in Brazil Regarding Mathematics Education", Mathematics Curriculum in School Education, 1, pp. 143-156.

[8] Suurtamm, C., Huntley, M. A. and Thompson, D. R. (2018) International Perspectives on Mathematics Curriculum, Information Age Publishing Inc., North Carolina.

[9] Macedo, E. (2006) "Currículo: Política, Cultura e Poder", Currículo sem Fronteiras, 6(2), pp. 98-113.

[10] Gomes, N. L. (2012) "Relações Étnico-Raciais, Educação e Descolonização dos Currículos", Currículo sem Fronteiras, 12(1), pp. 98-109.

[11] Berry, J. (2009) "Can There Be an Alternative to the Centralized Curriculum in England?", Improving Schools, 12(1), pp. 33-41.

[12] Aoki, T. (1986) "Teaching as In-dwelling Between Two Curriculum Worlds", The B.C. Teacher, Vancouver, 65(3).

[13] Valverde, G., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H. and Houang, R. T. (2002) "Textbooks and Educational Opportunity", in According to the Book, Springer Science + Business Media: New York, pp. 1-20.

[14] Rojano, T. and Solares-Rojas, A. (2018) "The mathematics curriculum design from an international perspective: methodological elements for a comparative analysis", in School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, University of Tsukuba: Tsukuba, pp. 475-482.

[15] OECD (2013) PISA 2012 Assessment and Analytical Framework: Mathematics, Reading, Science, Problem Solving and Financial Literacy, OECD, Paris.

[16] Simmt, E. (2017) "Curriculum in Canada: A Fractal Interpretation Using the Case of Alberta", in International Perspectives on Mathematics Curriculum, Information Age Publishing, pp. 103-130.

[17] Kilpatrick, J., Swafford, J. and Findell, B. (2001). Adding it up: Helping children learn mathematics, National Academy Press, Washington, DC.

[18] Sharma, S. (2013) "Qualitative Approaches in Mathematics Education Research: Challenges and Possible Solutions", Education Journal, 2(2), pp. 50-57.

[19] Zanette, M. (2017) "Qualitative research in the context of Education in Brazil", Educar em Revista, 65, pp. 149-166.

[20] Schubring, G. (2014) “A Matemática Elementar de um Ponto de Vista Superior: Felix Klein e a Sua Atualidade”, in O Saber do Professor de Matemática: Ultrapassando a Dicotomia entre Didática e Conteúdo, Ciência Moderna: Rio de Janeiro, pp. 39-54.

[21] Kilpatrick, J. (2008) “The Mathematics Teacher and Curriculum Change”, PNA, 3(3), pp. 107-121.

[22] Shulman, L. S. (1986) “Those who understand: Knowledge growth in teaching”, Educational Researcher, 15(2), 4-14.

[23] Wu, H. (2011) Understanding numbers in elementary school mathematics, American Mathematical Society Providence, RI.

[24] Behr, M. J., Lesh, R., Post, T.R. and Silver, E.A. (1983) “Rational number concepts”, in Acquisition of mathematics concepts and processes, Academic Press: New York, pp. 91-126.

[25] Son, J. W. (2012) “A cross-national comparison of reform curricula in Korea and the US in terms of cognitive complexity: the case of fraction addition and subtraction”, ZDM Mathematics Education, 44, pp. 161–174.

[26] Van de Walle, J. A., Karp, K. S. and Bay-Williams, J. M. (2009) Elementary and middle school mathematics: Teaching developmentally (7th ed.), Pearson Education, Boston.