The History of Mathematical Notation: An Inroad to Learning Mathematics

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Abstract

Teaching and learning mathematics as a language can help students broaden their perception of mathematics and help teachers assess that understanding. The part of mathematical language used the most by both student and teacher alike is symbolization often referred to as notation. Studying the history behind symbols, such as the equal sign and pi, gives students the opportunity to connect the concept to the symbol, thereby expanding their view and deepening their understanding of mathematics.

1. Introduction

When students are asked to define what mathematics is, they frequently offer "numbers" as an answer and seldom mention the concepts and processes behind those answers. In this era of rapidly progressing technology with quick answers, the need increases to find methods to help students broaden their perception of mathematics and to help teachers adequately assess that understanding. One of those techniques is to focus on mathematical language which Ball and Sleep [3] state "is both mathematical content to be learned and [a] medium for learning mathematical content" (p. 13). Viewing mathematics as a language shifts the focus from a simple numerical answer to the work shown through notation and makes the communication of mathematics synonymous with the demonstration of mathematical understanding. However, learning mathematics as a language has its own unique challenges.

2. Literature Review

According to Schleppegrell [12], a key challenge for mathematics teachers is to help students use their everyday language to move toward a better understanding and use of the technical language of mathematics. The implication is that the informal and formal languages can be connected to enhance student understanding of mathematical concepts. Looking to the historical development of mathematics as a language may help provide insight into that desired connection.

According to Burton [5], mathematics began with the need to count items and to record the results. Although the earliest cultures did not have the abstract symbols for numbers in common use now, they did have basic systems such as tallying to indicate quantities. As various societies grew and human interactions increased, the need for permanent records of transactions grew from simple finger tallying to recording marks on objects such as wood or stones. Burton describes this step towards permanence as one "not only toward abstract number concept, but also toward written communication" (p. 2). With Alexander the Great's invasion of India in antiquity, the mingling of Greek and Indian cultures "immensely stimulated the communication of ideas" [5, p. 227] between the two worlds. Burton states it is likely Indian mathematics in its earliest stages was influenced by the Greeks. Arguably, this influence is the result of an exchange of ideas that could not have occurred without some form of common mathematical language.

Like most languages, mathematical language evolved with the times to keep pace with a rapidly expanding world. From its earliest beginnings, mathematics was expressed in words to capture small quantities, such as the size of a herd, but as populations increased so did various quantities, resulting in the need for an efficient marking system that could be quickly used and recorded [5]. This pattern has been repeated throughout the history of mathematics in which a concept has been first articulated in words and then replaced with a symbol so that mathematicians could move more efficiently through their work.

In the 19^{th} century, German historian, Heinrich Nesselman described this evolution of a mathematical word into symbol as occurring in three stages: rhetorical, syncopated, and symbolic [13]. Silver illustrates these three stages by describing the history behind the use of the minus sign (-). For example, he notes that in medieval Europe the operation of subtraction was written as a word in various languages. For example, it was recorded as the word *minus* in Latin and *moins* in French. By the 15^{th} century, the word was syncopated or shortened to $\overline{m}$. By 1489, the syncopated version was being replaced with a symbol formed by deleting the *m* and leaving

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the – . In the same year, Johannes Widmann published an arithmetic book using the new symbol and compares it to the plus sign for addition, stating "What – is, that is less, and the + is more" [13, p. 369]. It can be argued that in comparing the two symbols, Widman is connecting the relationship between the fundamental mathematical concepts of subtraction or "less" and addition or "more" to the same relationship between the two symbols – i.e., that the minus sign has one mark less than the plus sign.

In the modern mathematics classroom, however, students frequently struggle with the symbols they are asked to use. Oyoo [10] states that the findings of his study reveal "that symbols affect the learning of mathematics" (p. 121) because, in part, students can have misconceptions about what the symbols mean. For example, Oyoo cites that some students in his study believed that the use of the equal sign simply meant that they had arrived at the answer. This superficial interpretation of the equal sign suggests the students did not have a full understanding of what it means for two quantities to be equal.

Schleppegrell [12] notes that if mathematics teachers do not first introduce concepts in ordinary language that students understand and then helps those students transition to properly using the technical language they need to develop and demonstrate a "full understanding of the concepts, [then] student learning suffers" (p. 156). Studying the rich history of the development of mathematical symbols helps with this transition by providing students with an opportunity to see the ordinary words that gave rise to the symbols. As Oyoo [10] states, "mathematical symbolization is … a process that gives meaning and relationships between mathematics ideas, objects, concepts, or processes…" (p. 119). Studying how those relationships were established historically provides an entry point for students to connect the symbol they are using to the concept they are being asked to learn. It brings into focus the evolution of mathematical language from simple spoken words and marks to the technical symbols now in use.

3. Discussion

Unlike other languages, the language of mathematics includes a multitude of diverse parts to include "symbols, oral language, written language, and visual representations such as graphs and diagrams" [12, p. 141]. It can be argued that the part of mathematical language used the most by both student and teacher alike is symbolization often referred to as notation. To effectively communicate about mathematics, students must be able to recognize, interpret, and translate mathematical notation. However, the use of mathematical notation can be confusing for students. Indeed, Mazur [8] notes that these "marks, signs, and symbols … are harder to learn than any natural language humans have ever created." (p. 18). To illustrate, notation can place multiple meanings within the use of one symbol. For example, parentheses are used in many different situations in mathematics. They can be used to indicate the order of operations, to note an ordered pair, to ask for the greatest divisor, to denote the input value of a function, etc. To help ease student confusion, research suggests that teachers need to use "better and more effective instructional strategies" [10, p. 128] to teach mathematical notation. One effective strategy may be found in providing the history of mathematical symbols to help students connect the notation to the concepts they are learning.

To contemplate the role history can play in teaching mathematical symbolization, educators must first embrace the notion that the purpose of notation is to encapsulate a substantial amount of information into a compacted form that can be easily manipulated. This encapsulation itself has been a historical journey that began with the use of words [13]. For example, in the development of algebra, Burton notes that prior to Diophantus' pivotal work *Arithmetica* in 250 CE, algebra was rhetorical, meaning that only words were used in problems and their solutions [5]. In *Arithmetica*, however, Diophantus paved the way to a consideration of a notation system by introducing his own shorthand notations for the work he was doing. This process has been called the "syncopation of algebra" [5, p. 221]. By the 16th century, the evolution of a universal system of notation for algebra entered a decisive phase with the work of French mathematician Viéta [5].

The historical transition from words to symbols in mathematics is also captured in the mental activity humans undergo when interpreting a mathematical symbol. For example, Mazur [8] states that the mind processes the symbol for a number by first associating it with things such as "three sheep in the field, three sheep … all the way up to "three-ness." Imagining physical objects decreases as generality increases. The mathematical symbol, therefore, is a visual anchor that helps the mind through the process of grasping the general through the particular" (p. 20). What this historical and mental process offers is that tracing a symbol back to its origin in words may provide students with a valuable opportunity to make connections with the mathematical concepts anchoring those symbols. What may be of particular interest to mathematics educators is the history behind some of the commonly used symbols of mathematics such as addition, division, congruence, pi, and equality.

One of the earliest operations children learn is addition along with its corresponding symbol, + , often called the "plus sign." History suggests that the "+" symbol derives from the "t" used in Latin manuscripts to shorten the word for "et" which translates as "and" [6, 8]. Oftentimes in introducing addition to young children, word problems are used such as "if you have three apples and I give you two, how many apples will you have?" The word "and" captures the addition in the question and the student will most likely use counting to obtain the answer. When taking the next step, however, of abstracting the process to the statement "3 + 2," some students may wonder about the use of what seems to be an arbitrary symbol. Although it is not necessary to teach young students Latin, it might nevertheless be helpful to them to learn that the symbol comes from a letter for the word "and" in another language. Learning that information gives meaning to the use of the plus sign and connects it to the mathematical concept of addition.

Another symbol introduced early in a child's education is the division symbol, ÷ , whose history reveals two different symbols combined into one notation. Ball [4] notes that "division was ordinarily denoted by the Arab way … in the form of a fraction by means of a line drawn between them in any of the forms a - b, a/b, or $\frac{a}{b}$ " (p. 199). In 1657, however, English mathematician William Oughtred [4] used the colon, : , to indicate a ratio. Ball contends that the modern symbol for division is a combination of the fraction symbol, – , from the Arab tradition and the colon used by Oughtred [4]. The power of this history is that it connects two mathematical concepts – division and fractions – into one symbol and reinforces the relationship between the two. This connection is particularly effective as an entry point to helping students learn about fractions as a division of the numerator by the denominator. This connection as represented in the division symbol and addressed by its history can help remove apprehension about fractions which students often view as a new and unrelated topic to anything they have previously learned.

Comparable to the history of the division symbol, the history of the geometric symbol for congruence, ≅ , also combines two mathematical ideas, shape and size, into one symbol. In geometry, if two figures have the same shape but are not necessarily the same size, they are classified as similar and the relationship between them is noted with the symbol ∼ . If the two figures are also the same size, that relationship is also indicated in the use of the = sign since their corresponding parts would have equal measures. Therefore, two figures that are congruent have both the same shape and the same size [2]. In the early eighteenth century, German mathematician, Christian von Wolf, was the first to use the symbol for similarity, ∼ , and equality, = , to note the concept of congruence, but he did not combine the symbols into one symbol. However, by the end of the 18th century, the combination of the two symbols into one symbol was being seen in texts [6]. Cajori [6] notes that the advantage of the combined symbol is that "it had the advantage of conveying more specifically the idea of congruence as the superposition of the ideas expressed by ∼ and = " (p. 415). Simply put, the congruence symbol captures the interplay of shape and size in establishing relationships between geometric figures.

Perhaps, the most popular geometric symbol is pi, π. In 1706, Welshman William Jones published a work in which the symbol, π, appeared for the first time to represent the ratio of the circumference of a circle to its diameter [9]. Jones understood that a precise number could not be given to express the ratio so he chose the Greek letter for p, π. It is thought he did so based upon the fact that π is the first letter of the Greek words for periphery and perimeter - two words that share the same prefix "peri" which means "all around" [9, 11]. By choosing a symbol based upon the notion of "around," Jones was likely connecting the definition of circumference as the distance around the circle to the symbol, π.

One of the most important and heavily used symbols in mathematics is the equal sign, and yet, it is frequently misused or avoided by students. The modern sign for equality, =, was first used in 1557 by Welsh physician, Robert Recorde, who indicated that he could not think of anything more equal than a pair of parallel lines [7]. What this history offers is a connection between the symbol and the concept of equality via a geometric implication. Parallel lines can be defined as two lines in the same plane that never intersect [2]. Because the lines are in the same plane and do not intersect, the distance between them remains the same at all points along the lines. Arguably, that "sameness" in the distance between the parallel lines underpins the concept of equality captured in the = symbol. Therefore, when the equal sign is used, what is written to the left and right of the sign should represent the same value.

4. Conclusion

Teaching mathematics from the perspective of a language has the potential to help students make connections with the mathematics they are being asked to learn, but there are challenges. These challenges are created in part out of the fact that a mathematics classroom is, in essence, a place where two languages blend – where a technical language merges with the common. These burdens present what Jill Adler [1] calls "dilemmas of mediation" (p. 235) in which teachers, in part, must balance "shaping informal, expressive and sometimes incomplete and confusing language, while aiming towards the abstract and formal language of mathematics" (p. 236). But a focus on the history behind the technical language of mathematical symbols can help students bridge the gap between the informal and formal language of mathematics. For example, highlighting that the addition symbol is a translation of the word "and" grounds both the concept and symbol in a common word frequently used and understood.

A focus on mathematical notation and its origins can also help teachers identify and increase the depth at which their students understand the mathematics they are learning. When students use mathematical notation in their work, it is sometimes difficult to tease out the difference between what is procedural and conceptual knowledge the students are demonstrating. Mathematics teachers sometimes desire to give students the benefit of the doubt and classify notation errors in student work as simple procedural mistakes. However, for example, if students struggle with the proper use of the equal sign, it should be taken as an indicator that they may have a superficial understanding of the concept of equality. Likewise, pi is often referred to as simply the number 3.14 with little thought to the ratio behind it. Examining the history behind mathematical symbols offers students the opportunity to explore the concepts they are being asked to learn by making connections to the notation they are being asked to use. Such experiences offer students the opportunity to deepen their understanding of mathematical concepts and broaden their perspective of mathematics as a language with a rich history.

5. References

[1] Adler, J. (1997). A participatory-inquiry approach and the mediation of mathematical knowledge in a multilingual classroom. Educational Studies in Mathematics, 33, 235–258.

[2] Alexander, D., & Koeberlein, G. (2011). Elementary geometry for college students (5th Ed.). Brooks/Cole.

[3] Ball, D., & Sleep, L. (2007). How does mathematical language figure in the work of teaching? How does this shape Mathematical Knowledge for Teaching (MKT) and teacher education? Presentation at the 11th Annual Meeting of the Association of Mathematics Teacher Educators, Irvine, CA.

[4] Ball, W.W. (2010). A short account of the history of mathematics (4th ed.). Dover Publications. (Original work published 1908)

[5] Burton, D. (2006). The history of mathematics: An introduction. McGraw-Hill.

[6] Cajori, F. (2011). A history of mathematical notations. Dover Publications. (Original work published 1929)

[7] Katz, V. (2009). A history of mathematics (6th ed.). Addison-Wesley.

[8] Mazur, J. (2014). Enlightening symbols: A short history of mathematical notation and its hidden powers. Princeton University Press.

[9] Merriam-Webster.com (n.d.). Peri- In Merriam-Webster.com dictionary. https://www.merriam-webster.com /dictionary/peri. (Access Date: 16 November 2024).

[10] Oyoo, P. O. (2022). Exploring students' proficiency in mathematical symbolization. International Journal of Advanced Research, 5(1), 117-130. DOI://doi.org/10. 37284/ijar.5.1.774.

[11] Roberts, G., (2015). Pi-Day 2015: meet the man who invented pi. https://www.theguardian.com/ science /alexs-adventures-in-numberland/2015/mar/14/pi-day-2015 - william-jones-the-welshman-who-invented-pi.

[12] Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. Reading & Writing Quarterly, 23, 139–159.

[13] Silver, D. (2017). The new language of mathematics. American Scientist, 105(6), 364-371. DOI: 10.1511/ 2017. 105.6.364.